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# Linear flow of heat in a semi-infinite-finite solid having a contact resistance

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Conduction of heat in a semi-infinite-finite solid having a contact resistance, has been worked out by the powerful operational method due to Heaviside. Expressions for temperature distribution in a finite and infinite solid are obtained. Special cases of a thin film attached to an infinite solid having wide applications in Engineering have also been worked out

## INTRODUCTION

In solving the problems of heat conduction through composite solids it is generally assumed that at the surface of separation the temperatures in the two media are the same This assumption will only be valid for very intimate contact, such as a soldered joint. In all other cases, even for optically flat surfaces pressed lightly together, the rate of heat transfer between the two surfaces is proportional to their temperature difference, (Carslaw & Jaeger 1959). The corresponding heat conduction problem in a semi-infinite-finite solid has not been fully discussed so far, to the best of our knowledge. Carslaw & Jaeger (1959) and Ghosh & Bhattacharya (1970) have studied similar problems but neglected the thermal resistance at the point of contact. In the present paper this problem including this thermal resistance has been worked out.

## EXPLANATIONS OF THE SYMBOL USED

$v_1$, $k_1$, $\rho_1$, $c_1$ and $h_1$ are the temperature, conductivity, density, specific heat and diffusivity respectively in the finite region, *i.e* in the range $-l < x < 0$, and $v_2$, $k_2$, $\rho_2$, $c_2$ and $h_2$ are the corresponding quantities in the infinite region, *i.e.* $x > 0$.

$x$ = variable length measured along the direction of propagation of heat flux.

$l$ = length of the finite region.

$V$ = temperature of the source at $x = -1$.

$t$ = variable time.

D = d/d$t$ (operator).

$H$ = coefficient of surface heat transfer.

## Solution of the Problem

The equations to be solved are

$$\frac{\partial^2 v_1}{\partial x^2} - \frac{1}{h_1}\frac{\partial v_1}{\partial t} = 0, \quad -l < x < 0, \quad t > 0 \qquad \dots (1)$$

$$\frac{\partial^2 v_2}{\partial x^2} - \frac{1}{h_2}\frac{\partial v_2}{\partial t} = 0, \quad x > 0, \quad t > 0 \qquad \dots (2)$$

The boundary conditions are

$$v_1 = V, \quad x = -l, \quad t > 0 \qquad \dots (3)$$

$$k_1 \frac{\partial v_1}{\partial x} = k_2 \frac{\partial v_2}{\partial x}, \; x = 0, \quad t > 0 \qquad \dots (4)$$

$$v_2 \to 0, \quad \text{as} \quad x \to \infty, \quad t > 0 \qquad \dots (5)$$

$$-k_1 \frac{\partial v_1}{\partial x} = H(v_1 - v_2), \; x = 0, \quad t > 0 \qquad \dots (6)$$

With initial temperature zero, equations (1) and (2) in operational form become

$$\frac{\partial^2 v_1}{\partial x^2} - \frac{\mathrm{D}}{h_1} v_1 = 0, \quad -l < x < 0 \qquad \dots (7)$$

$$\frac{\partial^2 v_2}{\partial x^2} - \frac{\mathrm{D}}{h_2} v_2 = 0, \quad x > 0 \qquad \dots (8)$$

writing $\mathrm{D}/h_1 = q_1^2$ and $\mathrm{D}/h_2 = q_2^2$ the equations (7) and (8) become

$$\frac{\partial^2 v_1}{\partial x^2} - q_1^2 v_1 = 0 \qquad \dots (9)$$

$$\frac{\partial^2 v_2}{\partial x^2} - q_2^2 v_2 = 0 \qquad \dots (10)$$

The solutions of equations (9) and (10) are

$$v_1 = A_1 \cosh q_1 x + B_1 \sinh q_1 x \qquad \dots (11)$$

$$v_2 = A_2 \cosh q_2 x + B_2 \sinh q_2 x \qquad \dots (12)$$

where $A_1$, $B_1$, $A_2$ and $B_2$ are arbitrary constants.

Evaluating the arbitrary constants from equations (3), (4), (5) and (6) the equations (11) and (12) turn out to be

$$v_1 = V\frac{\cosh q_1x-\sigma'\sinh q_1x}{\cosh q_1l+\sigma'\sinh q_1l} \quad \ldots (13)$$

$$v_2 = \frac{VH\exp(-q_2x)}{(H+q_2k_2)\cosh q_1l+\sigma'\sinh q_1l}, \quad \ldots (14)$$

where

$$\sigma' = \sigma\Big/\left(1+\frac{q_2k_2}{H}\right)$$

and

$$\sigma = \frac{k_2q_2}{k_1q_1} = \left(\frac{k_2\rho_2c_2}{k_1\rho_1c_1}\right)^{\frac{1}{2}}, \text{ a constant.}$$

Expanding the hyperbolic sines and cosines and simplifying,

$$v_1 = V\sum_{n=0}^{\infty}\left[(m)^n\exp\left(-\frac{D^{\frac{1}{2}}}{h_1^{\frac{1}{2}}}\{x+(2n+1)l\}\right)\right.$$

$$\left.-(m)^{n+1}\exp\left(-\frac{D^{\frac{1}{2}}}{h_1^{\frac{1}{2}}}\{(2n+1)l-x\}\right)\right] \quad \ldots (15)$$

$$v_2 = \frac{2VH}{H_1+q_2k_2}\cdot\sum_{n=0}^{\infty}(m)^n\exp\left[-D^{\frac{1}{2}}\left\{\frac{x}{h_2^{\frac{1}{2}}}+\frac{(2n+1)l}{h_1^{\frac{1}{2}}}\right\}\right] \quad \ldots (16)$$

where $m = (\sigma'-1)/(\sigma'+1)$ and $H_1 = H(1+\sigma)$.

The operational solutions of equations (15) and (16) are given by

$$v_1 = V\left[a_{nn_1n_2}(4h_1t)^{N/2}(i)^N\operatorname{erfc}\left\{\frac{x+(2n+1)l}{2(h_1t)^{\frac{1}{2}}}\right\}\right.$$

$$\left.-a_{n+1n_1n_2}(4h_1t)^{N/2}(i)^N\operatorname{erfc}\left\{\frac{(2n+1)l-x}{2(h_1t)^{\frac{1}{2}}}\right\}\right] \quad \ldots (17)$$

$$v_2 = 2VHb_{nn_1n_2}(4h_1t)^{N'/2}(i)^{N'}\operatorname{erfc}\left\{\frac{xk+(2n+1)l}{2(h_1t)^{\frac{1}{2}}}\right\} \quad \ldots (18)$$

where $k = (h_1/h_2)^{\frac{1}{2}}$,

$$a_{nn_1n_2} = \frac{(-1)^{n+n_1+n_2}(nc_{n_1})H^{n_1}{}_2H^{n_2}{}_1\{n_1(n_1+1)\ldots(N-1)\}}{n_2\,!\,k_2^{n_2}}(h_2/h_1)^{N/2}, \quad \ldots (19)$$

$$a_{n+1n_1n_2} = \frac{(-1)^{n+1^{n_1n_2}}({}^{n+1}c_{n_1})H_2^{n_1}H_1^{n_2}\{n_1(n_1+1)\ldots(N-1)\}}{n_2\,!\,k_2^{n_2}}(h_2/h_1)^{N/2}, \quad \ldots (20)$$

and

$$b_{nn_1n_2} = \frac{(-1)^{n+n_1+n_2}({}^nc_{n_1})H_2^{n_1}H_1^{n_2}\{(n_1+1)(n_2+2)\ldots(N\}}{n_2\,!\,k_2^{n_2+1}}(h_2/h_1)^{N'/2} \quad \ldots (21)$$

For (19), (20) and (21),

$$\begin{aligned} n &= 0, 1, 2, \ldots, \infty \\ n_2 &= 0, 1, 2, \ldots, \infty \\ N &= n_1+n_2 \\ H_2 &= 2H\sigma/k_2 \\ N' &= N+1 \end{aligned}$$

for (19) and (21),

$$n_1 = 0, 1, 2, \ldots, n,$$

and for (20) only,

$$n_1 = 0, 1, 2, \ldots, (n+1).$$

The temperature gradient at the surface is found to be

$$\left(\frac{\partial v_1}{\partial x}\right)_{x=-l} = -V\, q_1\left\{1+2\sum_{n=1}^{\infty}(m)^n \exp(-2nq_1 l)\right\} \qquad . (22)$$

$$\left(\frac{\partial v_1}{\partial x}\right)_{x=-l} = -V\left[\frac{1}{(\pi h_1 t)^{\frac{1}{2}}} + a_{nn_1n_2}\left\{(4h_1t)^{(N-1)/2}(i)^{N-1}\operatorname{erfc}\frac{nl}{(h_1t)^{\frac{1}{2}}}\right\}\right] \qquad (23)$$

The equation (23) can be used for a correct estimate of the age of the earth.

## Special Cases

*Case I*

When $H$ tends to infinity. $\sigma' = \sigma$ is a constant and $m = (\sigma-1)/(\sigma+1)$ is also a constant.

The operational solutions of (15) and (16) turn out as

$$v_1 = V\sum_{n=0}^{\infty}(m)^n\left\{\operatorname{erfc}\frac{(2n+1)l+x}{2(h_1t)^{\frac{1}{2}}} - (m)\operatorname{erfc}\frac{(2n+1)l-x}{2(h_1t)^{\frac{1}{2}}}\right\} \qquad . (24)$$

$$v_2 = \frac{2V}{1+\sigma}\sum_{n=0}^{\infty}(m)^n.\ \operatorname{erfc}\left\{\frac{kx+(2n+1)l}{2(h_1t)^{\frac{1}{2}}}\right\} \qquad . (25)$$

Again, the temperature gradient at the surface is given by

$$\begin{aligned}\left(\frac{\partial v_1}{\partial x}\right)_{x=-l} &= -V[q_1\{1+2\sum_{n=1}^{\infty}(m)^n \exp(-2nq_1l)\}] \\ &= -\frac{V}{(\pi h_1 t)^{\frac{1}{2}}}\left[1+2\sum_{n=1}^{\infty}(m)^n \exp\left(-\frac{n^2l^2}{h_1t}\right)\right] \end{aligned} \qquad (26)$$

For large value of time $\exp(-n^2l^2/h_1t) \approx 1$, and we have

$$\left(\frac{\partial v_1}{\partial x}\right)_{x=-l} = -\frac{V}{(\pi h_1 t)^{\frac{1}{2}}}\{1+2m(1+m+m^2+\ldots)\}$$

$$= -\frac{V}{(\pi h_1 t)^{\frac{1}{2}}}\left(\frac{k_2\rho_2 c_2}{k_1\rho_1 c_1}\right)^{\frac{1}{2}} \quad \ldots (27)$$

Carslaw & Jaeger (1959) used the equation to calculate the age of the earth Taking the case of granite and air as the composition of earth and the surrounding thin film of air, the quantity $\left(\frac{k_2\rho_2 c_2}{k_1\rho_1 c_1}\right)^{\frac{1}{2}}$ comes out to be nearly 450

*Case II(A)*

When $l$ is small, *i.e.*, when a thin film of another substance is attached to the end of a semi-infinite medium, we have from euquation (14).

$$v_2 = V.\frac{\exp(-q_2x)}{1+q_2(k_2/H+1/h)}$$

$$= V\frac{\exp(-q_2x)}{1+q_2/h'} \quad \ldots (28)$$

where

$$h = (k_1/k_2 l)$$

and

$$h' = \frac{Hh}{k_2h+H}$$

The operational solution of the equation (28) will be

$$v_2/V = \operatorname{erfc}\frac{x}{2(h_2t)^{\frac{1}{2}}} - \exp(h'x+h_2th'^2)\operatorname{erfc}\left\{\frac{x}{2(h_2t)^{\frac{1}{2}}}+h'(h_2t)^{\frac{1}{2}}\right\} \quad \ldots (29)$$

*Case II(B)*

When $l$ is small and $H$ tends to infinity, we have $h = h'$ and the equation (29) turn out as

$$v_2/V = \operatorname{erfc}\frac{x}{2(h_2t)^{\frac{1}{2}}} - \exp(hx+h_2th^2)\operatorname{erfc}\left\{\frac{x}{2(h_2t)^{\frac{1}{2}}}+h(h_2t)^{\frac{1}{2}}\right\} \quad \ldots (30)$$

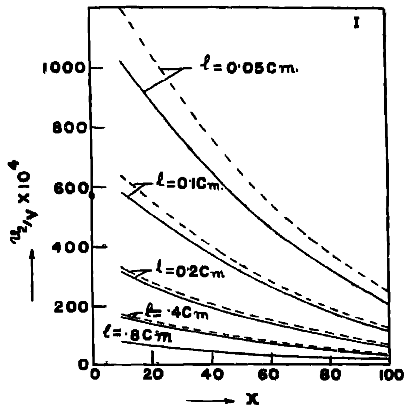
Equations (29) and (30) are computed by using the following data :—

$k_1 = 0{\cdot}0001$, $h_1 = 0{\cdot}0014$

$k_2 = 0{\cdot}93$, $h_2 = 1{\cdot}14$,

$H = 0{\cdot}01$, $t = 1$ hour,

Two graphs are drawn : (*i*) distance *vs* temperature (using the equations (29) & 30)) in figure 1 and (*ii*) film thickness *vs.* temperature (using the equation (29)) in figure 2.




1. Temperature in the infinite region is plotted against distance $x$ in cm for different values of film thickness $l$. Equation (29) is represented by full curves, and the equation (30) is represented by dashed curves.

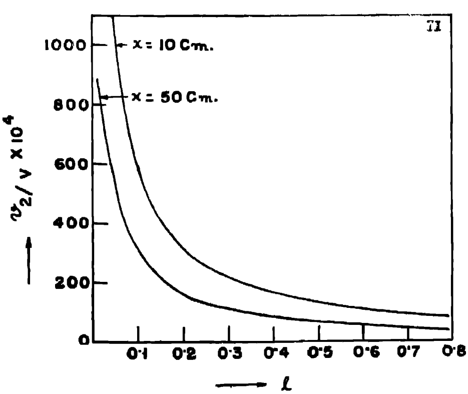



Figure 2. Temperature in the infinite region is plotted against film thickness $l$ in cm for different values of distance $x$,

## Discussion

Equations (24), (25), (27) are obtained by Carslaw & Jaeger (15959), and the equation (30) is obtained by Ghosh & Bhattacharya (1970) foı the linear flow of heat in semi-infinite-finite solid having no contact resistance. What is, when $H$ tends to infinity, contact resistance has no effect on the temperature distribution in the finite or infinite region.

From figure 1 it is clear that at large film thickness the temperature in the infinite region is affected slightly, but in case of small film thickness the temperature distribution in the infinite medium is very prominent. But for any large value of $x$ the temperature distribution is only slightly affected by film thickness It is also obvious from figure 1 that for a particular value of $x$, an increase in the film thickness decreases the effect of contact resistance on the temperature distribution in the infinite medium.

From figure 2 it is obvious that for any particular value of $x$, an increase in the film thickness decreases the value of temperature At low value of thickness the temperature rapidly falls to a lower value The nature of this graph is same as obtained by Ghosh & Bhattacharya (1970).

Further work on the heat flow in concentric spheres in presence of contact resistance is under consideration. Thanks are due to Prof. A. Sinha for his encouragement

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